Basic properties of homomorphisms

Suppose $\phi: G \to H$ is a homomorphism.

1a) $\phi(e_G) = e_H$

Pf: $\forall x \in G$, $\quad \phi(x) = \phi(x \cdot e_G)$ (def. of iden.)

$= \phi(x) \cdot \phi(e_G)$ (ϕ is a hom.)

(cancellation law)

$\Rightarrow \phi(e_G) = e_H$. ∎

(inverse in G)

1b) $\forall g \in G$, $\quad \phi(g^{-1}) = \phi(g)^{-1}$

(inverse in H)

Pf: $\phi(g^{-1})\,\phi(g) \underset{\phi \text{ is a hom.}}{=} \phi(g^{-1}g) = \phi(e_G) \underset{1a}{=} e_H$

$\Rightarrow \phi(g^{-1}) = \phi(g)^{-1}$. ∎

1c) $\forall g \in G$, $n \in \mathbb{Z}$, $\quad \phi(g^n) = \phi(g)^n$.

Pf: ϕ is a hom. + induction + 1b ... ∎

2) $\phi(G) \leq H$ (image of a hom. is a subgroup of codomain)

Pf: Use subgroup criteria. $\phi(G)$ is:

- Non-empty ✓ $\phi(e_G) \in \phi(G)$

- Closed under multiplication ✓

  If $h_1, h_2 \in \phi(G)$ then $\exists\, g_1, g_2 \in G$ s.t. $\phi(g_1) = h_1$, $\phi(g_2) = h_2$.

  Then $g_1 g_2 \in G$, so $h_1 h_2 = \phi(g_1)\phi(g_2) = \phi(g_1 g_2) \in \phi(G)$.
  (ϕ is a hom.)

- Closed under inverses ✓

  If $h \in \phi(G)$, write $h = \phi(g)$ for some $g \in G$.

  Then $g^{-1} \in G$ and $\phi(g^{-1}) = \phi(g)^{-1}$.
  1b

Therefore $\phi(G) \leq H$. ∎

Ex: $G = H = \mathbb{Z}$, $\phi: G \to H$, $\phi(k) = nk$ $(n \in \mathbb{Z})$

ϕ is a hom. $\Rightarrow \phi(G) = n\mathbb{Z} = \{nk : k \in \mathbb{Z}\} \leq \mathbb{Z}$.

(kernel of a hom. is a subgroup of domain)

the <u>kernel</u> of ϕ

3) Define $\ker(\phi) = \{ g \in G : \phi(g) = e_H \}$. Then $\ker(\phi) \leq G$.

Pf: Subgroup crit. $\ker(\phi)$ is:

- Non-empty ✓ $\phi(e_G) = e_H \Rightarrow e_G \in \ker(\phi)$.

- Closed under multiplication ✓

  If $g_1, g_2 \in \ker(\phi)$ then $\phi(g_1 g_2) \overset{\phi \text{ is a hom.}}{=} \phi(g_1)\phi(g_2) \overset{g_1, g_2 \in \ker(\phi)}{=} e_H \cdot e_H = e_H$

  $\Rightarrow g_1 g_2 \in \ker(\phi)$.

- Closed under inverses ✓

  If $g \in \ker(\phi)$ then $\phi(g^{-1}) \overset{1b}{=} \phi(g)^{-1} \overset{g \in \ker(\phi)}{=} e_H^{-1} = e_H \Rightarrow g^{-1} \in \ker(\phi)$.

Therefore $\ker(\phi) \leq G$. $\blacksquare$

Ex: $G = (GL_2(\mathbb{R}), \cdot)$, $H = (\mathbb{R} \setminus \{0\}, \cdot)$, $\phi : G \to H$, $\phi(A) = \det(A)$.

$\ker(\phi) = \{ A \in G : \phi(A) = \underbrace{e_H}_{1} \}$

$= \{ A \in GL_2(\mathbb{R}) : \det(A) = 1 \} = SL_2(\mathbb{R}) \leq G$.

4) If ϕ is an isomorphism then its inverse function $\phi^{-1}: H \to G$ is also an isomorphism.

Pf: ϕ^{-1} is a bijection, so we just need to show that it is a homomorphism. Suppose $h_1, h_2 \in H$, $\phi^{-1}(h_1) = g_1$, $\phi^{-1}(h_2) = g_2$.

Then $\phi(g_1) = h_1$ and $\phi(g_2) = h_2$ (def. of inv. fn.)

$\Rightarrow h_1 h_2 = \phi(g_1)\phi(g_2) = \phi(g_1 g_2)$ (ϕ is a hom.)

$\Rightarrow \phi^{-1}(h_1 h_2) = g_1 g_2 = \phi^{-1}(h_1)\phi^{-1}(h_2)$ (def. of inv. fn.) $\square$

Ex: $G = (\mathbb{R}, +)$, $H = (\mathbb{R}_{>0}, \cdot)$, $\phi: G \to H$, $\phi(x) = e^x$.

ϕ is an isom. $\Rightarrow \phi^{-1}: \mathbb{R}_{>0} \to \mathbb{R}$ is an isom.

Note: $\phi^{-1}(x) = \log x$, so $\phi^{-1}(xy) = \phi^{-1}(x) + \phi^{-1}(y)$

$\Rightarrow \log(xy) = \log x + \log y$.